The perfect foresights' assumption revisited: (I) the existence of equilibrium with multiple price expectations

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Abstract

Our earlier papers [2,3,4,5,6] had extended to asymmetric information the classical existence theorems of general equilibrium theory [1,7,10], under the standard assumption that agents had perfect foresights, that is, they knew, ex ante, which price would prevail on each spot market.

Common observation suggests, however, that agents more often trade with an unprecise knowledge of future prices. We now let agents anticipate, in each random state, a <u>*set*</u> *of plausible spot prices and introduce a mild condition of consistent anticipations and a related concept of 'correct foresights equilibrium', along which the 'true' spot prices are, ex ante, in all agents' anticipations. We prove, in a finite economy with standard assumptions, that existence of such equilibrium is still characterized by the no-arbitrage condition of finance. This result, which extends our earlier theorems, shows that private information and uncertain anticipations would not affect existence, but the* <u>*value*</u> *of equilibrium prices and allocations.*

Key words: general equilibrium, incomplete markets, asymmetric information, arbitrage, existence of equilibrium.

JEL Classification: D52

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1 Introduction

When asymmetric information prevails on financial markets, the existence and value of equilibrium prices and allocations depend crucially on how agents update their beliefs from observing markets. In a traditional response, called '*rational expectations*', agents would refer, quoting Radner, 1979, [11], to "*a 'model' or 'expectations' of how equilibrium prices are determined*".

In our approach [2,3], agents having asymmetric information and no price expectations along Radner would update their beliefs from analysing arbitrage on financial markets. A broad concept of equilibrium would embed, as two particular applications, the classical financial equilibrium, and equilibrium with asymmetric information. Assessing the existence properties of that equilibrium, we showed in [4,5,6] that the classical results of symmetric information, e.g., Cass, 1984 [1], for nominal assets, Geanakoplos-Polemarchakis, 1986 [10], for numeraire assets, Duffie-Shafer, 1985 [7], for real assets, would extend to asymmetric information, namely, that a financial equilibrium (or '*pseudo-equilibrium*' in case of real assets) would always exist under a no-arbitrage condition, whatever agents' information.

So far, we had retained the standard assumption that agents believed with certainty which commodity price would prevail tomorrow, conditionnally on the random state. This single price expectation, better known as '*perfect foresights*' when all expectations are correct, is at the core of general equilibrium concepts, including, Arrow-Debreu's and Radner's sequential equilibrium, Grandmont's temporary equilibrium, and Radner's rational expectations equilibrium with asymmetric information. Yet, common observation suggests agents more often trade with no precise knowledge (ex ante) of the spot price which will prevail after nature has played.

We now introduce the simplest setting to treat that problem, namely, a two-period model with a finite set of anticipations in each state and a probability distribution over expected prices by every agent (the case of continuous price expectations and arbitrary probability distributions is dealt with in a next paper). We present the related concept of competitive equilibrium, which allows agents to be uncertain about future spot prices. This concept of equilibrium coincides with that of our earlier model [4] whenever agents anticipate exactly one price in each state. It replaces the condition of perfect foresights by a milder one of '*correct foresights*', along which the '*true*' spot prices (i.e., those which can prevail tomorrow) are, ex ante, in all agents' expectations. We define a notion of '*structure of beliefs*', which leads to correct foresights by letting agents have, at least, one common anticipation on each spot market. Starting, for simplicity, with nominal assets, we define the related notion of equilibrium, or '*correct foresights equilibrium*' (C.F.E.). We show that the existence of this equilibrium is characterized, as in the classical theory, by the no-arbitrage condition, whatever agents' private information and structure of beliefs. This result embeds and extends our earlier existence theorem [4] to the case of multiple expectations. It shows that private information or anticipations would not affect existence, but the value, of equilibrium prices and allocations. It may explain phenomena such as equity premium, speculation, bubbles on markets, which are a puzzle to the classical theory, where agents perfectly anticipate prices.

Section 2 presents the basic model, where agents may have asymmetric information and finitely many price expectations in each future state. It introduces the notions of structure of payoffs, information and beliefs and of correct foresights equilibrium. It states the existence Theorem. Section 3 builds on a standard fixed-point-like argument in a finite economy to prove this Theorem.

2 The basic model

We consider a pure-exchange financial economy with two periods ($t \in \{0,1\}$). The economy is finite, in the sense that the sets of agents, $I := \{1,...,m\}$, of commodities, $\{1,...,L\}$, states of nature, S, and assets, $\{1,...,J\}$, are all finite. There is an a priori uncertainty at the first period ($t = 0$) about which state $s \in S$ will prevail at the second period ($t = 1$). Throughout, we shall denote by $s = 0$ the non-random state at $t = 0$ and let $\Sigma_0 := \{0\} \cup \Sigma$, for each subset Σ of S.

2.1 Information and markets

At the first period, each agent $i \in I$ receives or infers a private information signal, in the form of a discrete probability θ_i over S, and we denote by θ^i (with upper subscript) the agent's '*information set*', namely, $\theta^i := \{s \in S : \theta_i(s) > 0\}$. Henceforth, we let $\theta := (\theta_i)_{i=1}^m$ and $\underline{\boldsymbol{\theta}} := \cap_{i=1}^m \theta_i$ be agents' pooled information set. We assume that $\underline{\boldsymbol{\theta}}$ always contains the "true state", i.e., the state which will prevail at $t = 1$. Under this assumption, we show in [3] that agents updating their information from analysing arbitrage opportunities on financial markets never reduce their information sets beyond $\underline{\boldsymbol{\theta}}$, hence, remain correctly informed in the sense that they always expect the true state to be realizable.

Along all information available at $t = 0$, every state $s \in \underline{\boldsymbol{\theta}}$ (and only those states) may prevail at $t = 1$. At a competitive equilibrium, for each $s \in \underline{\boldsymbol{\theta}}$, there exists one spot price $p[s] \in \mathbb{R}^L$, at which commodities will be traded if state s obtains.[2]

[2] Throughout the paper, we denote by $\cdot$ and $\|.\|$, respectively, the scalar product and Euclidean norm. For every $\widetilde{\Sigma} \subset S_0 := \{0\} \cup S$ and every $\Sigma \subset \widetilde{\Sigma}$, for every $S \times J$-matrix $V := (v_j[s])_{(s,j) \in S \times J}$ and every $\Sigma \times J$-matrix A, for all set $X \subset \mathbb{R}^{L\widetilde{\Sigma}}$ and all set $X' \subset \mathbb{R}^{\widetilde{\Sigma}}$, for all collections $(q, s, l) \in \mathbb{R}^J \times \Sigma \times \{1,...,L\}$ and

At $t = 0$, each agent $i \in I$ observes the spot price $p[0] \in \mathbb{R}^L$ of commodities, also denoted p_0, the asset price $q \in \mathbb{R}^J$, but needs not be aware of future spot prices, as in the traditional approach of equilibrium with perfect foresights. Agents' information at $t = 0$ is thus characterized by the collection (θ, p_0, q). This yields the following notions of information structure and market price.

Definition 1 *A collection $\theta := (\theta_i) := (\theta_i)_{i=1}^m$ of m probability distibutions over S, such that $\underline{\boldsymbol{\theta}} := \cap_{i=1}^m \{s \in S : \theta_i(s) > 0\} \neq \varnothing$, is called an (information) structure and Θ denotes their set.*

We let $\theta \in \Theta$ stand for $\theta := (\theta_i) \in \Theta$, and, given $(\theta, i, s) \in \Theta \times I \times S$, $\theta_{(i,s)} := \theta_i(s)$ be the i^{th} agent's probabilty assessment that state s prevails, $\theta^i := \{s \in S : \theta_{(i,s)} > 0\}$ be her information set and $\theta_0^i := \{0\} \cup \theta^i$, $\underline{\boldsymbol{\theta}} := \cap_{i=1}^m \theta^i$ be the pooled information set and $\underline{\boldsymbol{\theta}}_0 := \{0\} \cup \underline{\boldsymbol{\theta}}$. We say that $\theta \in \Theta$ is symmetric if $\theta^i = \theta^j$, for every $(i, j) \in I^2$.

Given $\theta \in \Theta$, a market price is a vector $p(\theta) \in \mathbb{R}^{L\underline{\boldsymbol{\theta}}_0}$. Their set (a subset of $\mathbb{R}^{L\underline{\boldsymbol{\theta}}_0}$) is denoted by $\mathcal{M}(\theta)$. Given $\theta \in \Theta$ and $p \in \mathcal{M}(\theta)$, the vectors $p[s] \in \mathbb{R}^L$, for each $s \in \underline{\boldsymbol{\theta}}_0$, are referred to as the spot prices (in state s).

$(x, x', (y, y'), (z, z')) \in \mathbb{R}^{L\widetilde{\Sigma}} \times \mathbb{R}^{\widetilde{\Sigma}} \times (\mathbb{R}^{\Sigma})^2 \times (\mathbb{R}^{L\Sigma})^2$, we denote by:
1) $x[\Sigma]$ and $x'[\Sigma]$, respectively, the truncations of x on $\mathbb{R}^{L\Sigma}$ and x' on $\mathbb{R}^{\Sigma}$;
2) $X[\Sigma] := \{x \in \mathbb{R}^{L\Sigma} : \exists \widetilde{x} \in X, x = \widetilde{x}[\Sigma]\}$;
3) $X'[\Sigma] := \{x' \in \mathbb{R}^{\Sigma} : \exists \widetilde{x} \in X', x = \widetilde{x}[\Sigma]\}$;
4) $A[s]$, $y[s]$, $z[s]$, resp., the row, scalar, vector, indexed by $s \in \Sigma$, of A, y, z;
5) $z^l[s]$ the l^{th} component of $z[s] \in \mathbb{R}^L$ and let $z^l := (z^l[s]) \in \mathbb{R}^{\Sigma}$;
6) $y \leqslant y'$ and $z \leqslant z'$ (resp. $y << y'$ & $z << z'$) the relationships $y[s] \leqslant y'[s]$ and $z^l[s] \leqslant z'^l[s]$ (resp. $y[s] < y'[s]$ and $z^l[s] < z'^l[s]$) for every $(l, s) \in \{1, ..., L\} \times \Sigma$;
7) $y < y'$ (resp. $z < z'$) the relationships $y \leqslant y'$, $y \neq y'$ (resp. $z \leqslant z'$, $z \neq z'$);
8) $z_\square z'$ the vector $(z[s] \cdot z'[s]) \in \mathbb{R}^{\Sigma}$, $y_\square z$ the vector $(y[s]z[s]) \in \mathbb{R}^{L\Sigma}$;
9) $V(\Sigma)$ (when $0 \notin \Sigma$) the $\Sigma \times J-$matrix s.t. $V(\Sigma)[s] := V[s]$, for each $s \in \Sigma$;
10) $W(\Sigma, q)$ (when $0 \notin \Sigma$) the $\Sigma_0 \times J-$matrix such that $W(\Sigma, q)[0] := -q$ and, for each $s \in \Sigma$, $W(\Sigma, q)[s] := V[s]$ and we let $W(q) := W(S, q)$;
11) $\mathbb{R}^{L\Sigma}_+ := \{x \in \mathbb{R}^{L\Sigma} : x \geqslant 0\}$, $\mathbb{R}^{\Sigma}_+ := \{x \in \mathbb{R}^{\Sigma} : x \geqslant 0\}$,
$\mathbb{R}^{L\Sigma}_{++} := \{x \in \mathbb{R}^{L\Sigma} : x >> 0\}$, $\mathbb{R}^{\Sigma}_{++} := \{x \in \mathbb{R}^{\Sigma} : x >> 0\}$.

For each $(i,\theta) \in I \times \Theta$, the i^{th} agent has for consumption set $X_i(\theta) := \mathbb{R}_+^{L\theta_0^i}$, and an endowment, $e_i(\theta) \in X_i(\theta)$, satisfying the following standard Assumption:

Assumption S1 *(strong survival):* $\forall (i,\theta) \in I \times \Theta, e_i(\theta) >> 0$.

By the end of the first period, agents will have reached some (final) information structure $\theta \in \Theta$ and made their trade and consumption plans, jointly on two markets: the commodity market and the financial market.

The commodity market consists, given $\theta \in \Theta$, in $\#\underline{\boldsymbol{\theta}}_0$ spot markets for commodities, which agents consume or exchange, at $t = 0$, on the spot market of the non-random state $s = 0$, and, at $t = 1$, on the spot market of the particular state $s \in \underline{\boldsymbol{\theta}}$, which will prevail. A market price, $p \in \mathcal{M}(\theta)$, embeds the $\#\underline{\boldsymbol{\theta}}_0$ spot prices $p[s] \in \mathbb{R}^L$ (for $s \in \underline{\boldsymbol{\theta}}_0$) at which commodities would be traded on each state-s spot market. In a competitive economy, this price is unique.

The financial market permits limited transfers across periods and states, via J nominal assets $j \in \{1, ..., J\}$, whose contingent payoffs, in each state $s \in S$, are denoted by $v_j[s]$ and yield a $S \times J-$matrix $V = (v_j[s])_{(s,j) \in S \times \{1,...,J\}}$, assumed to be fixed and of full column-rank ($J = rankV$). We denote by $\mathcal{V}(S, J)$ the set of $S \times J$ matrixes and also refer to the pair $[V, \theta] \in \mathcal{V}(S, J) \times \Theta$ as a structure.

With no loss of generality, agents are not endowed in assets and may exchange portfolios unrestrictively at $t = 0$. A portofolio $z := (z^j) \in \mathbb{R}^J$ specifies the quantity z^j of each asset $j \in \{1, ..., J\}$, positive, if purchased, or negative, if sold. Given $\theta \in \Theta$ and the asset price $q \in \mathbb{R}^J$, an agent $i \in I$ may thus purchase a portofolio $z := (z^j) \in \mathbb{R}^J$ for $q \cdot z$ units of account at $t = 0$, against the expected flow $V(\theta^i)z$ of payoffs at $t = 1$.

One particular class of structures $[V, \theta] \in \mathcal{V}(S, J) \times \Theta$ is of interest: those for which there exists a price $q \in \mathbb{R}^J$ and a collection of wheights $(\lambda_i) \in \Pi_{i=1}^m \mathbb{R}_{++}^{\theta^i}$, such that

$q = {}^t\lambda_i V(\theta^i)$, for each $i \in I$. Then, by a standard separation argument, the financial market grants no agent an arbitrage at price q, that is, the possibility of a positive money transfer, in one state, at no cost in any other. When that condition holds, the structure $[V, \theta]$ is said to be (q-)arbitrage-free.

In the remainder of the paper, a fixed structure $[V, \theta] \in \mathcal{V}(S, J) \times \Theta$ is given and assumed to represent agents' final information and transfer opportunities at $t = 0$. Unless stated otherwise, the reference to $V \in \mathcal{V}(S, J)$ and $\theta \in \Theta$ applies to that structure and is dropped in all above definitions and notations. With no loss of generality, we assume that $S = \cup_{i=1}^m \theta^i$.

2.2 Consumer's expectations and behavior

Ex ante, agents need not know future spot prices. The spot prices they anticipate need not be unique in each given state, as they have been, so far, in general equilibrium models, whether Grandmont's temporary equilibrium, Radner's rational expectations equilibrium, or the classical models of perfect foresights.

Instead, we assume that agents have (at $t = 0$) a set of plausible prices for the second period ($t = 1$), called '*price expectations*', and an idiosyncratic probability distribution over such anticipated prices. To simplify presentation, we limit ourselves, in this introductory paper, to the case of discrete probabilities and a fixed number of expectations in each state for each agent. This finite dimensional economy is a much simpler framework to start with and prove the existence of equilibrium (than one with arbitrary price expectations sets), because the standard fixed-point-like arguments of Euclidean spaces apply. In a companion paper, extending the model to the infinite setting, we will use arbitrary price sets and probability measures.

In any sequential equilibrium, all agents' expectations (at $t = 0$) must include the actual spot prices (at $t = 1$), since any of these spot prices may randomly prevail tomorrow. Thus, given $p \in \mathcal{M}$, agents, observing $p[0]$ on the non-random spot market, will not consider the possibility of another spot price at $t = 0$, but they may be led to agree (e.g., by a shared view, or by chance), rightly or wrongly, that each $p[s] \in \mathbb{R}^L$ (for $s \in \underline{\boldsymbol{\theta}}$) is one possibility for the spot price in state s tomorrow. When this agreement takes place, and when p turns out to be the market price of equilibrium, agents are said to have '*correct price foresights*'. Indeed, correct foresights coincide with perfect foresights whenever agents have exactly one price expectation in each state of their information set.

To a fictitious observer (informed of all agents' beliefs), the price expectations of agents having at least one expectation in common on each spot market, can be represented by a so-called '*structure of beliefs*', along the following Definition.

Definition 2 *A structure of beliefs is a collection* $\pi := (\pi_i)$ *of* m *mappings* $\pi_i : \mathcal{M} \to \Pi_i$ *(where* Π_i *denotes the set of discrete probabilities on* $\mathbb{R}^{L\theta_0^i}$*), such that, for every* $(i, p) \in I \times \mathcal{M}$*, the three following Conditions hold:*

(a) *the probability* $\pi_i(p) \in \Pi_i$*, also denoted* $\pi_{(i,p)}$*, is the product of* $\#\theta_0^i$ *independent probabilities* $\pi_{(i,p,s)}$ *on* $\mathbb{R}^L$ *(one for each* $s \in \theta_0^i$*), i.e.,* $\pi_{(i,p)}(\overline{p}) := \otimes_{s \in \theta_0^i} \pi_{(i,p,s)}(\overline{p}[s])$*, for every* $\overline{p} \in \mathbb{R}^{L\theta_0^i}$*;*

(b) *for each* $s \in \theta_0^i$*, the* i^{th} *agent's set of "spot price" expectations in state* s*, namely,* $\mathcal{P}_{(\pi,i,p,s)} := \{\overline{p}[s] \in \mathbb{R}^L : \pi_{(i,p,s)}(\overline{p}[s]) > 0\}$*, contains a fixed number* $n_{(i,s)} \in \mathbb{N}^*$ *of elements (i.e., independent of* p*), and we let* $\mathcal{P}_{(\pi,i,p,s)} := \{p^1_{(\pi,i,p,s)}, ..., p^{n_{(i,s)}}_{(\pi,i,p,s)}\}$*;*

(c) *the* i^{th} *agent's set of price expectations,* $\mathcal{P}_{(\pi,i,p)} := \{\overline{p} \in \mathbb{R}^{L\theta_0^i} : \pi_{(i,p)}(\overline{p}) > 0\}$*, satisfies* $p \in \mathcal{P}_{(\pi,i,p)}[\underline{\boldsymbol{\theta}}_0]$ *and* $\mathcal{P}_{(\pi,i,p)}[0] := \mathcal{P}_{(\pi,i,p,0)} := \{p_0 := p[0]\}$*. We let* $n_i := \otimes_{s \in \theta_0^i} n_{(i,s)} = \#P_{(\pi,i,p)}$ *and* $\mathcal{P}_{(\pi,i,p)} := \{p^1_{(\pi,i,p)}, ..., p^{n_i}_{(\pi,i,p)}\}$*.*

We denote by Φ *the set of structures of beliefs.*

The notion of structure of beliefs embeds the case of perfect foresights of our previous model [4], in which, for each $(i,p) \in I \times \mathcal{M}$, the price expectations set $\mathcal{P}_{(\pi,i,p)}$ reduces to a singleton $\{p_i\}$, such that $p_i[\underline{\boldsymbol{\theta}}_0] := p$. Henceforth, we assume that agents are endowed with a structure of beliefs $\pi \in \Phi$, which is fixed and dropped in all notations, and we refer to the following Assumptions on π, where, for every $(r,\overline{p}) \in \mathbb{R}_{++} \times \mathbb{R}^L$, $B^r(\overline{p})$ stands for the open ball of radius r of $\mathbb{R}^L$ centered on $\overline{p}$.

Assumption B1 *(continuous beliefs):* $\forall (i,p,\varepsilon,r) \in I \times \mathcal{M} \times \mathbb{R}_{++} \times \mathbb{R}_{++}, \exists \eta \in \mathbb{R}_{++}:$ $(p' \in \mathcal{M}$ *and* $\|p'-p\| < \eta) \Rightarrow (p^k_{(i,p',s)} \in B^r(p^k_{(i,p,s)})$ *and* $|\pi_{(i,p,s)}(p^k_{(i,p,s)}) - \pi_{(i,p',s)}(p^k_{(i,p',s)})| < \varepsilon$, *for every* $s \in \theta^i_0$ *and every* $k \in \{1,...,n_{(i,s)}\})$.

Assumption B2 *(bounded beliefs):* $\exists (\gamma,\delta) \in \mathbb{R}^2_{++} : \delta |p^l[s]| \leqslant |\overline{p}^l[s]| \leqslant \gamma |p^l[s]|$,
$\delta \min_{s \in \underline{\boldsymbol{\theta}}} |p^l[s]| \leqslant |\overline{p}^l[\overline{s}]| \leqslant \gamma \sup_{s \in \underline{\boldsymbol{\theta}}} |p^l[s]|$ *and* $\delta \min_{s \in \underline{\boldsymbol{\theta}}} \|p[s]\| \leqslant \|\overline{p}[\overline{s}]\| \leqslant \gamma \sup_{s \in \underline{\boldsymbol{\theta}}} \|p[s]\|$,
$\forall (i,p,l) \in I \times \mathcal{M} \times \{1,...,L\}, \forall \overline{p} \in \mathcal{P}_{(i,p)}, \forall (s,\overline{s}) \in \underline{\boldsymbol{\theta}} \times \theta^i \backslash \underline{\boldsymbol{\theta}}$.
Moreover, $\overline{p}[s] \geqslant 0, \forall (i,p) \in I \times \mathcal{M}, \forall s \in \theta^i, \forall \overline{p}[s] \in \mathcal{P}_{(i,p,s)} \backslash \{p[s]\}$.

In [4], we proved that, under perfect foresights, the absence of arbitrage opportunities on financial markets characterized the existence of equilibrium, under the same standard assumptions, whether agents had symmetric or asymmetric information. We now extend this result to a refined notion of equilibrium, where agents, endowed with a structure of beliefs, may only have correct price foresights, that is, $p \in \cap_{i=1}^m \mathcal{P}_{(i,p)}[\underline{\boldsymbol{\theta}}_0]$ for the market price $p \in \mathcal{M}$ of equilibrium.

2.3 The notion of correct foresights equilibrium (C.F.E.)

For each $i \in I$, let $Y_i := \Pi_{s \in \theta^i_0} (\mathbb{R}^L_+)^{n_{(i,s)}}$ be the i^{th} agent's set of consumption plans, whose economic interpretation is as follows: given a market price $p \in \mathcal{M}$ (which exists along agents' beliefs, but needs not be known by them) and a state $s \in \theta^i_0$, the i^{th} agent's consumption plan $y \in Y_i$ relates each price expectation $\overline{p}[s] \in \mathcal{P}_{(i,p,s)}$, to a conditional (on the realization of $\overline{p}[s]$) consumption vector $y(\overline{p}[s]) \in \mathbb{R}^L_+$ on the state-s market. Then, $y := (y(\overline{p}))_{\overline{p} \in \mathcal{P}_{(i,p)}}$, where, for every $\overline{p} \in \mathcal{P}_{(i,p)}$, $y(\overline{p}) := (y(\overline{p}[s]))_{s \in \theta_0^i}$ is defined from above.[3] This notion of consumption plans embeds and extends the classical one, in which $p \in \mathcal{M}$ is known by all agents and consumption decisions in each state are, therefore, unique.

Given market prices $(p,q) \in \mathcal{M} \times \mathbb{R}^J$, we define, for each $i \in I$:

$$B_i(p,q) := \{(y,z) \in Y_i \times \mathbb{R}^J : \overline{p}[s] \cdot (y(\overline{p}[s]) - e_i[s]) \leqslant W(q)[s] \cdot z, \forall (\overline{p}, s) \in \mathcal{P}_{(i,p)} \times \theta_0^i\},$$

as the i^{th} agent's budget set. An allocation is a collection $y := (y_i) \in Y := \Pi_{i=1}^m Y_i$ of consumption plans. It is said to be (p-)attainable whenever $\sum_{i=1}^m (y(p[s]) - e_i[s]) = 0$, for each $s \in \underline{\boldsymbol{\theta}}_0$ (we recall that $p \in \mathcal{P}_{(i,p)}[\underline{\boldsymbol{\theta}}_0]$).

With a slight abuse, we henceforth denote $y(p) := (y(p[s]))_{s \in \underline{\boldsymbol{\theta}}_0} \in X_i[\underline{\boldsymbol{\theta}}_0]$, for every $(i,p) \in I \times \mathcal{M}$ and every $y \in Y_i$; whereas $y(\overline{p})$, for $\overline{p} \in \mathcal{P}_{(i,p)}$, stands for $y(\overline{p}) := (y(\overline{p}[s]))_{s \in \theta_0^i} \in X_i$. Then, the above conditions are written: $\sum_{i=1}^m (y_i(p) - e_i[\underline{\boldsymbol{\theta}}_0]) = 0$. Thus, the sets of attainable allocations, portfolios and strategies at market prices $(p,q) \in \mathcal{M} \times \mathbb{R}^J$ are, respectively:

$\mathcal{A}(p) := \{(y_i) \in Y : \sum_{i=1}^m (y_i(p) - e_i[\underline{\boldsymbol{\theta}}_0]) = 0\} := \{(y_i) \in Y : \sum_{i=1}^m (y_i(p[s]) - e_i[s]) = 0, \forall s \in \underline{\boldsymbol{\theta}}_0\}$,
$\mathcal{Z} := \{(z_i) \in (\mathbb{R}^J)^m : \sum_{i=1}^m z_i = 0\}$ and
$\mathcal{A}(p,q) := \{[(y_i, z_i)] \in \Pi_{i=1}^m B_i(p,q) : (y_i) \in \mathcal{A}(p), (z_i) \in \mathcal{Z}\}$.

Given $p \in \mathcal{M}$, the generic i^{th} agent needs not know the market price p and has beliefs represented by $\pi_{(i,p)}$. She is endowed with a conditional (on $\pi_{(i,p)}$) utility function, $U_{(i,p)} : Y_i \to \mathbb{R}_+$, and preference correspondence, $\Gamma_{(i,p)} : Y_i \to Y_i$, defined by

[3] For each $i \in I$ and each $s \in \theta_0^i$, the fixed number $n_{(i,s)}$ of price expectations in Definition 2 and Assumption *B1* permit to assign (for each $k \in \{1, ..., n_{(i,s)}\}$) a fixed place to the conditional consumption $y(p^k_{(i,p,s)}) \in \mathbb{R}^L_+$ in Y_i when $p \in \mathcal{M}$ varies.

$\Gamma_{(i,p)}(y) := \{y' \in Y_i : U_{(i,p)}(y') > U_{(i,p)}(y)\}$, for all $y \in Y_i$, meeting the following Condition:

Assumption S2 *(separability):* $\forall i \in I, \forall s \in \theta^i, \forall k \in \{1, ..., n_{(i,s)}\}, \exists u^k_{(i,s)} : \mathbb{R}^2_+ \to \mathbb{R}_{++}$, *s.t.* $\forall p \in \mathcal{M}, \forall y \in Y_i$, $U_{(i,p)}(y) = \sum_{(s,k)\in\theta^i\times\{1,...,n_{(i,s)}\}} \theta_{(i,s)}\pi_{(i,p,s)}(p^k_{(i,p,s)})u^k_{(i,s)}(y(p_0), y(p^k_{(i,p,s)}))$.

The economy described above for a given triple $[V, \theta, \pi] \in \mathcal{V}(S, J)\times\Theta\times\Phi$, also called structure, is denoted by $\mathcal{E}_{[V,\theta,\pi]}$. An equilibrium of this economy is the collection of prices $(p, q) \in \mathcal{M} \times \mathbb{R}^J$ and strategies, which are p-attainable and optimal for every agent in the budget set. Under perfect foresights, it coincides with the notion of no-arbitrage equilibrium of [4].

Definition 3 *Let a structure of payoffs, information and beliefs* $[V, \theta, \pi] \in \mathcal{V}(S, J)\times\Theta\times\Phi$ *and a corresponding economy* $\mathcal{E}_{[V,\theta,\pi]}$ *be given. A collection of prices,* $(p, q) \in \mathcal{M} \times \mathbb{R}^J$*, and strategies,* $[(y_i, z_i)] \in \Pi^m_{i=1}B_i(p, q)$*, defines an equilibrium of the economy* $\mathcal{E}_{[V,\theta,\pi]}$*, or 'correct foresights equilibrium' (C.F.E.), if:*

(a) $\forall i \in I,\ B_i(p, q) \cap \Gamma_{(i,p)}(y_i) \times \mathbb{R}^J = \varnothing$;

(b) $(y_i) \in \mathcal{A}(p)$;

(c) $(z_i) \in \mathcal{Z}$.

To prove the existence of equilibrium, we shall refer to the following Assumptions. An economy $\mathcal{E}_{[V,\theta,\pi]}$, which meets the Conditions of Assumptions *S1–S2-B1-B2* above, and *A1-A2-A3* on preferences, below, will be called '*standard*'.

Assumption A1 *(monotonicity):* $U_{(i,p)}(y') > U_{(i,p)}(y), \forall(i, p) \in I\times\mathcal{M}, \forall(y, y') \in Y^2_i,\ y'>y$.

Assumption A2 *(continuity):* $\forall(i, p) \in I \times \mathcal{M}, \forall y \in Y_i, \forall\varepsilon > 0, \exists\eta > 0 :$ $(y' \in Y_i, \|y' - y\| < \eta) \Rightarrow (|U_{(i,p)}(y') - U_{(i,p)}(y)| < \varepsilon)$.

Assumption A3 *(quasi-concavity):* $U_{(i,p)}(y + \lambda(y' - y)) \geqslant \min(U_{(i,p)}(y), U_{(i,p)}(y'))$, $\forall(i, p) \in I \times \mathcal{M}, \forall((y, y'), \lambda) \in Y^2_i\times]0, 1[$, *with a strict inequality if* $U_{(i,p)}(y) \neq U_{(i,p)}(y')$.

Remark 1: Assumptions *S1–A1-A2-A3* are standard and result in a strictly positive market price $p \in \mathcal{M}$ at equilibrium. Assumption *B2* conveys the idea that agents have non-negative price expectations, which cannot be indefinitely bigger or smaller than their common expectations on spot markets. A separability Assumption, such as *S2*, is standard and technical. Indeed, the joint Assumptions *B2-S2* will permit to bound the economy in the equilibrium problem (see the Appendix).

We now state the existence theorem, which embeds and extends the existence characterization of equilibrium by the no-arbitrage condition, as demonstrated in [4] under perfect foresights, to the case of the correct foresights equilibrium.

Theorem 1 *Let a structure* $[V, \theta, \pi] \in \mathcal{V}(S, J) \times \Theta \times \Phi$ *and a standard economy* $\mathcal{E}_{[V,\theta,\pi]}$ *be given. The economy* $\mathcal{E}_{[V,\theta,\pi]}$ *admits a (correct foresights) equilibrium if, and only if,* $[V, \theta]$ *is arbitrage-free. Moreover, any equilibrium price system* $(p, q) \in \mathcal{M} \times \mathbb{R}^J$ *is such that* $p >> 0$ *and* $[V, \theta]$ *is* q*-arbitrage-free.*

Before proving that every standard economy $\mathcal{E}_{[V,\theta,\pi]}$ with an arbitrage-free structure $[V, \theta]$ admits an equilibrium, Claim 1 provides a converse result.

Claim 1 *In a standard economy* $\mathcal{E}_{[V,\theta,\pi]}$*, if a collection of prices* $(p, q) \in \mathcal{M} \times \mathbb{R}^J$ *and strategies* $[(y_i, z_i)] \in \Pi_{i=1}^m B_i(p, q)$ *satisfy Condition* (a) *of Definition 3 of equilibrium, then,* $p >> 0$ *and* $[V, \theta]$ *is* q*-arbitrage-free, hence, arbitrage-free.*

Proof Let a standard economy $\mathcal{E}_{[V,\theta,\pi]}$, prices $(p, q) \in \mathcal{M} \times \mathbb{R}^J$ and strategies $[(y_i, z_i)] \in \Pi_{i=1}^m B_i(p, q)$ be given, which satisfy Condition (a) of Definition 3.

Assume, by contraposition, that $p^{\bar{l}}[\bar{s}] \leqslant 0$, for some $(\bar{l}, \bar{s}) \in \{1, ..., L\} \times \underline{\boldsymbol{\theta}}_0$. Then, $x_1 \in Y_1$, defined by $x_1^{\bar{l}}(p[\bar{s}]) := 1 + y_1^{\bar{l}}(p[\bar{s}])$, by $x_1^{\bar{l}}(\overline{p}[\bar{s}]) := y_1^{\bar{l}}(\overline{p}[\bar{s}])$, for every $\overline{p} \in \mathcal{P}_{(1,p)}$, such that $\overline{p}[\bar{s}] \neq p[\bar{s}]$, and by $x_1^l(\overline{p}[s]) := y_1^l(\overline{p}[s])$, for every $(\overline{p}, (l, s)) \in \mathcal{P}_{(1,p)} \times (\{1, ..., L\} \times \theta_0^1 \backslash (\bar{l}, \bar{s}))$, satisfies, from Assumption $A1$, $(x_1, z_1) \in B_1(p,q) \cap \Gamma_{(1,p)}(y_1) \times \mathbb{R}^J$, which contradicts the fact that (y_1, z_1) meets Definition 3-(a). This proves $p >> 0$.

Assume, now, that $W(\theta^i, q)z > 0$, for some $(i,z) \in I \times \mathbb{R}^J$ (i.e., $[V,\theta]$ fails to be q-arbitrage-free). Let $\overline{s} \in \theta_0^i$ be such that $W(\theta^i, q)[\overline{s}] \cdot z > 0$. Then, by the same token, we let the reader check that there exists $x_i \in Y_i$, such that $(x_i, z_i + z) \in B_i(p,q)$ and, for every $\overline{p} \in \mathcal{P}_{(i,p)}$, $x_i(\overline{p}[\overline{s}]) > y_i(\overline{p}[\overline{s}])$ and $x_i(\overline{p}[s]) := y_i(\overline{p}[s])$, for each $s \in \theta_0^i \backslash \{\overline{s}\}$. Then, $(x_i, z_i + z) \in B_i(p,q) \cap \Gamma_{(i,p)}(y_i) \times \mathbb{R}^J$, from Assumption $A1$, which contradicts the fact that (y_i, z_i) meets Definition 3-(a). Hence, $[V,\theta]$ is q-arbitrage-free. $\square$

3 The existence theorem

Henceforth, an arbitrage-free structure $[V,\theta]$ and a standard economy $\mathcal{E}_{[V,\theta,\pi]}$ are set as given. First, we bound strategies to apply a fixed-point argument.

3.1 Bounding the economy

The following Lemmas will serve to bound agents' strategies in the equilibrium problem. As is now standard from [4], we let, for each $i \in I$:

$Z_i^o := \{z \in \mathbb{R}^J : V[s] \cdot z = 0, \forall s \in \theta^i\}$ and denote by $Z_i^{o\perp}$ its orthogonal;

$Z^o := \sum_{i=1}^m Z_i^o$ and denote by $Z^{o\perp} = \cap_{i \in I} Z_i^{o\perp}$ its orthogonal.

We consider the following compact sets:

$\mathcal{M} := \{p \in \mathbb{R}^{L\underline{\boldsymbol{\theta}}_0} : \|p[s]\| \leqslant 1, \textit{ for every } s \in \underline{\boldsymbol{\theta}}_0\}$;

$\mathcal{M}^* := \{p \in \mathcal{M} \cap \mathbb{R}_+^{L\underline{\boldsymbol{\theta}}_0} : \|p[s]\| = 1, \textit{ for every } s \in \underline{\boldsymbol{\theta}}\}$;

$\mathcal{M}_\varepsilon := \{p \in \mathcal{M} : p^l[s] \geqslant \varepsilon, \textit{ for every } (l,s) \in \{1,...,L\} \times \underline{\boldsymbol{\theta}}\}, \textit{ for every } \varepsilon \in]0, \frac{1}{L}[$;

$Q := \{q \in Z^{o\perp} : \|q\| \leqslant 1\}$;

$\Pi := \mathcal{M} \times Q$, $\Pi^* := \mathcal{M}^* \times Q$, $\Pi_\varepsilon := \mathcal{M}_\varepsilon \times Q$, $\textit{for every } \varepsilon \in]0, \frac{1}{L}[$.

Then, for every $(i,(p,q)) \in I \times \Pi$, we let:

$\overline{B}_i(p,q) := \{(y,z) \in Y_i \times Z_i^{o\perp} : \overline{p}[s] \cdot (y(\overline{p}[s]) - e_i[s]) \leqslant W(q)[s] \cdot z + 1, \forall (\overline{p},s) \in \mathcal{P}_{(i,p)} \times \theta_0^i\}$;

$\overline{\mathcal{A}}(p,q) := \{[(y_i,z_i)] \in \Pi_{i=1}^m \overline{B}_i(p,q) : (y_i) \in \mathcal{A}(p), \sum_{i=1}^m z_i \in Z^o\}$.

For any collections $(p,q) \in \Pi$ and $[(y_i,z_i)] \in \Pi_{i=1}^m \overline{B}_i(p,q)$, we denote $y := (y_i)$, $y(p) := (y_i(p))$, $z := (z_i)$, $(y,z) := [(y_i,z_i)]$. We now state Lemma 1.

Lemma 1 *The following Assertions hold:*

(i) $\exists r > 0 : (p \in \mathcal{M}$ and $y \in \mathcal{A}(p)) \Rightarrow \|y(p)\| < r$;

(ii) $\forall \varepsilon \in]0, \frac{1}{L}[, \exists r_\varepsilon > 0 : ((p,q) \in \Pi_\varepsilon,\ (y,z) \in \overline{\mathcal{A}}(p,q)) \Rightarrow (\|y\| + \|z\| < r_\varepsilon)$.

Proof see the Appendix. □

We now let $\Delta := \cap_{s \in \underline{\boldsymbol{\theta}}} \Delta_s$, where, for each $s \in \underline{\boldsymbol{\theta}}$:

$\Delta_s := \{p \in \mathcal{M}^* : \exists i \in I, \exists y := (y_i) \in \mathcal{A}(p)$, *such that:* $(x_i \in \Gamma_{(i,p)}(y_i),\ x_i(\overline{p}[\overline{s}]) = y_i(\overline{p}[\overline{s}])$, *for every* $(\overline{p},\overline{s}) \in \mathcal{P}_{(i,p)} \times \theta_0^i \backslash \{s\}$ *and* $x_i(\overline{p}[s]) = y_i(\overline{p}[s])$, *for every* $\overline{p}[s] \in \mathcal{P}_{(i,p,s)} \backslash \{p[s]\})$ *imply* $(p[s] \cdot x_i(p[s]) \geqslant p[s] \cdot y_i(p[s]) \geqslant p[s] \cdot e_i[s])\}$.

Lemma 2 *The following Assertions hold:*

(i) *For every* $s \in \underline{\boldsymbol{\theta}}$, Δ_s *is compact, hence* Δ *is a compact set;*

(ii) *There exists* $\underline{\varepsilon} \in]0, \frac{1}{L}[$, *such that* $\Delta \subset \mathcal{M}_{\underline{\varepsilon}}$.

Proof see the Appendix. □

The above Lemmas serve to bound the economy. We now set as given $\underline{\varepsilon} > 0$, which satisfies Lemma 2-(ii), and and $r := r_{\underline{\varepsilon}}$, which satisfies both upper-bound Conditions of Lemma 1 relative to $\underline{\varepsilon}$, and, for every $(i,(p,q)) \in I \times \Pi$, we let:

$Y_i^* := \{y \in Y_i : \|y(\overline{p}[s])\| \leqslant r, \forall (\overline{p},s) \in \mathcal{P}_{(i,p)} \times \theta_0^i\}$ and $Y^* := \Pi_{i=1}^m Y_i^*$;

$Z_i^* := \{z \in Z_i^{o\perp} : \|z\| \leqslant r\}$;

$\Gamma^*_{(i,p)} : Y^* \to Y_i^*$ be defined by $\Gamma^*_{(i,p)}(y) := \Gamma_{(i,p)}(y_i) \cap Y_i^*$ for every $y := (y_i) \in Y^*$;

$B_i^*(p,q) := B_i(p,q) \cap Y_i^* \times Z_i^* \subset \overline{B}_i(p,q)$;

$\mathcal{A}^*(p) := Y^* \cap \mathcal{A}(p) := \{(y_i) \in Y^* : \sum_{i=1}^m (y_i(p) - e_i[\underline{\boldsymbol{\theta}}_0]) = 0\}$.

3.2 The existence proof

Following [4,8], we define, for all $(i,(p,q)) \in I \times \Pi$, the modified budget sets:

$$B_i'(p,q) := \{(y,z) \in Y_i^* \times Z_i^* : \overline{p}[s] \cdot (y(\overline{p}[s]) - e_i[s]) \leqslant W(q)[s] \cdot z + \gamma_{(p,q)})[s], \forall (\overline{p}, s) \in \mathcal{P}_{(i,p)} \times \theta_0^i\};$$

$$B_i''(p,q) := \{(y,z) \in Y_i^* \times Z_i^* : \overline{p}[s] \cdot (y(\overline{p}[s]) - e_i[s]) << W(q)[s] \cdot z + \gamma_{(p,q)})[s], \forall (\overline{p}, s) \in \mathcal{P}_{(i,p)} \times \theta_0^i\},$$

where $\gamma_{(p,q)} \in [0,1]^{S_0}$ is defined as follows:

$\gamma_{(p,q)}[0] := 1 - \min(1, \|p[0]\| + \|q\|)$,

$\gamma_{(p,q)}[s] := 1 - \|p[s]\|$ for every $s \in \underline{\boldsymbol{\theta}}$

and $\gamma_{(p,q)}[\overline{s}] := 1 - \min_{s \in \underline{\boldsymbol{\theta}}} \|p[s]\|$ for every $\overline{s} \in S \backslash \underline{\boldsymbol{\theta}}$.

Claim 2 *For every* $(i,(p,q)) \in I \times \Pi$, $B_i''(p,q) \neq \emptyset$.

Proof Let $(i,(p,q)) \in I \times \Pi$ be given.

From Assumption *S1*, we can choose $y \in Y_i^*$ such that, $(\overline{p}_\square[y(\overline{p})\text{-}e_i]) \leqslant 0$, for every $\overline{p} \in \mathcal{P}_{(i,p)}$, with a strict inequality in any state $s \in \theta_0^i$ such that $\overline{p}[s] \neq 0$. Moreover, we let the reader check, from the definition of $\gamma_{(p,q)}$ and from Assumption *B2*, that the above plan y may be chosen such that $(\overline{p}_\square[y(\overline{p}) - e_i])[s] := \overline{p}[s] \cdot (y(\overline{p}[s]) - e_i[s]) < \gamma_{(p,q)}[s]$ holds for every $(\overline{p}, s) \in \mathcal{P}_{(i,p)} \times \theta^i$.

Assume, first, that $p[0] \neq 0$ or $\|p[0]\| + \|q\| < 1$. Then, from above, $(y, 0) \in B_i''(p,q)$. Assume, alternatively, that $p[0] = 0$ and $\|q\| = 1$. Then, from above, for $\eta > 0$ small enough, $(y, -\eta q) \in B_i''(p,q)$, since $q \in Z^{o\perp} \backslash \{0\} \subset Z_i^{o\perp} \backslash \{0\}$. $\square$

Claim 3 *For every $i \in I$, B_i'' is convex-valued and lower semicontinuous.*

Proof Let $(i,(p,q)) \in I \times \Pi$ be given. The convexity of $B_i''(p,q)$ is immediate. From Assumptions *B1-S2-A2* (and footnote 3), B_i'' has obviously an open graph in a compact set and, as a standard corollary, is lower semicontinuous. □

Claim 4 *For every $i \in I$, B_i' is convex-valued and upper semicontinuous.*

Proof For each $i \in I$, B_i' is obviously non-empty convex-valued, and upper semicontinuous, as is standard, for having a closed graph in a compact set. □

We now introduce an additional agent, $i = 0$, representing the market and, following Gale-Mas-Colell, 1975-1979 [9], a reaction correspondence for each agent, defined on the convex compact set $\Theta := \Pi \times (\Pi_{i=1}^{m} Y_i^* \times Z_i^*)$. Thus, we let, for each $i \in I$ and every $((p,q),(y,z) := [(y_i,z_i)]) \in \Theta$:

$$\Psi_i((p,q),(y,z)) := \left\{ \begin{array}{ll} B_i'(p,q) & \text{if } (y_i,z_i) \notin B_i'(p,q) \\ B_i''(p,q) \cap \Gamma^*_{(i,p)}(y) \times Z_i^* & \text{if } (y_i,z_i) \in B_i'(p,q) \end{array} \right\}.$$

And we define the market reaction correspondence as follows:

$$\Psi_0((p,q),(y,z)) := \{(p',q') \in \Pi : (p'-p) \cdot \textstyle\sum_{i=1}^{m} (y_i(p) - e_i[\underline{\boldsymbol{\theta}}_0]) + (q'-q) \cdot \sum_{i=1}^{m} z_i > 0\}.$$

Claim 5 *For each $i \in \{0,1,...,m\}$, Ψ_i is lower semicontinuous.*

Proof First, the correspondence Ψ_0 is lower semicontinuous for having an open graph. Second, we let $i \in I$ and $\omega := ((p,q),(y,z)) \in \Theta$ be given and consider separately the two alternatives $(y_i,z_i) \notin B_i'(p,q)$ and $(y_i,z_i) \in B_i'(p,q)$ and show that, in both cases, Ψ_i is lower semicontinuous at ω.

- Assume, first, that $(y_i,z_i) \notin B_i'(p,q)$. Then, $\Psi_i(\omega) = B_i'(p,q)$.

Let V be an open set in $Y_i^* \times Z_i^*$, such that $V \cap B_i'(p,q) \neq \emptyset$. It follows from the convexity of $B_i'(p,q)$ and the non-emptyness of the open set $B_i''(p,q)$ that $V \cap B_i''(p,q) \neq \emptyset$. From Claim 3, there exists a neighborhood U of (p,q) in Π, such that $V \cap B_i'(p',q') \supset V \cap B_i''(p',q') \neq \emptyset$, for every $(p',q') \in U$.

Since $B_i'(p,q)$ is nonempty, closed, convex in the compact set $Y_i^* \times Z_i^*$, there exist two open sets V_1 and V_2 in $Y_i^* \times Z_i^*$, such that $(y_i, z_i) \in V_1$, $B_i'(p,q) \subset V_2$ and $V_1 \cap V_2 = \emptyset$. From Claim 4, there exists a neighborhood $U_1 \subset U$ of (p,q), such that $B_i'(p',q') \subset V_2$, for every $(p',q') \in U_1$. Let $W = U_1 \times \Pi_{j=1}^m W_j$, where $W_i := V_1$ and $W_j := Y_j^* \times Z_j^*$, for every $j \in I \backslash \{i\}$. Then, W is a neighborhood of ω, such that $\Psi_i(\omega') = B_i'(p',q')$, and, from above, $V \cap \Psi_i(\omega') \neq \emptyset$, for every $\omega' := ((p',q'),(y',z')) \in W$. This proves the lower semicontinuity of Ψ_i at ω.

- Assume, now, that $(y_i, z_i) \in B_i'(p,q)$, i.e., $\Psi_i(\omega) = B_i''(p,q) \cap \Gamma^*_{(i,p)}(y) \times Z_i^*$.

The lower semicontinuity of Ψ_i at ω is immediate if $\Psi_i(\omega) = \emptyset$. Assume $\Psi_i(\omega) \neq \emptyset$. We let the reader check, from Assumptions *B1-S2-A2*, that both correspondences B_i'' and $(p',y') \in \mathcal{M} \times Y^* \mapsto \Gamma^*_{(i,p')}(y')$ have open graphs. As a standard corollary, $((p',q'),(y',z')) \in \Theta \mapsto B_i''(p',q') \cap \Gamma^*_{(i,p')}(y') \times Z_i^* \subset B_i'(p',q') \subset Y_i^* \times Z_i^*$, which has an open graph, is lower semicontinuous at ω. From the definition and latter inclusions in a compact set, Ψ_i is lower semicontinuous at ω. $\square$

Claim 6 *There exists* $(p^*,q^*),(y^*,z^*)) \in \Theta$, *such that:*

(i) $\forall (p,q) \in \Pi, (p^* - p) \cdot \sum_{i=1}^m (y_i^*(p^*) - e_i[\underline{\boldsymbol{\theta}}_0]) + (q^* - q) \cdot \sum_{i=1}^m z_i^* \geqslant 0;$

(ii) $\forall i \in I,\ (y_i^*, z_i^*) \in B_i'(p^*,q^*)$ *and* $B_i''(p^*,q^*) \cap \Gamma^*_{(i,p^*)}(y^*) \times Z_i^* = \emptyset$.

Proof Quoting Gale-Mas-Colell, 1975-79 [9,10]: "Given $X = \Pi_{i=1}^m Xi$, where X_i is a non-empty compact convex subset of an Euclidean space, let $\varphi_i : X \to X_i$ be m

convex (possibly empty) valued lower semicontinuous correspondences. Then, there exists x in X such that for each i either $x_i \in \varphi_i(x)$ or $\varphi_i(x) = \emptyset$".

The correspondences $\Psi_0 : \Theta \to \Pi$, $\Psi_i : \Theta \to X_i^* \times Z_i^*$ (for $i \in I$) satisfy the conditions of the above theorem, hence, admit an element $\omega^* := ((p^*, q^*), [(y_i^*, z_i^*)]) \in \Theta$, s.t., either $(p^*, q^*) \in \Psi_0(\omega^*)$, or $\Psi_0(\omega^*) = \emptyset$, and, for each $i \in I$, either $(y_i^*, z_i^*) \in \Psi_i(\omega^*)$, or $\Psi_i(\omega^*) = \emptyset$. By construction, $(p^*, q^*) \notin \Psi_0(\omega^*)$ and, for each $i \in I$, $(y_i^*, z_i^*) \notin \Psi_i(\omega^*)$, since $y_i^* \notin \Gamma^*_{(i,p^*)}(y^*)$. Then, $\Psi_0(\omega^*) = \emptyset$ yields Assertion (i) and $\Psi_i(\omega^*) = \emptyset$, for each $i \in I$, yields Assertion (ii). □

Claim 7 $\sum_{i=1}^m z_i^* \in Z^o$.

Proof Assume, by contraposition, $\sum_{i=1}^m z_i^* \notin Z^o$. Then, from Claim 6-$(i)$, $q \cdot \sum_{i=1}^m z_i^* \leqslant q^* \cdot \sum_{i=1}^m z_i^*$, for every $q \in Q := \{q \in Z^{o\perp} : \|q\| \leqslant 1\} \supset \{0\}$, which implies $q^* \cdot \sum_{i=1}^m z_i^* > 0$ and $\|q^*\| = 1$, hence, $\gamma_{(p^*,q^*)}[0] = 0$.

From Claim 6-(ii), $(y_i^*, z_i^*) \in B_i'(p^*, q^*)$, for each $i \in I$, whose budget constraint in state $s = 0$ is written: $p^*[0] \cdot (y_i^*(p^*[0]) - e_i[0]) \leqslant -q^* \cdot z_i^*$.

Summing up, for each $i \in I$, yields, from above:

$$p^*[0] \cdot \sum_{i=1}^m (y_i^*(p^*[0]) - e_i[0]) \leqslant -q^* \cdot \sum_{i=1}^m z_i^* < 0.$$

This contradicts Claim 6-(i), which implies $p^*[0] \cdot \sum_{i=1}^m (y_i^*(p^*[0]) - e_i[0]) \geqslant 0$ (take $(p[0], p[\underline{\boldsymbol{\theta}}], q) = (0, p^*[\underline{\boldsymbol{\theta}}], q^*)$ in Claim 6-(i)). □

Remark 2 Claim 7 has two consequences. First, $Q \subset Z^{o\perp}$ and Claim 6-(i) yields: $\forall p \in \mathcal{M}$, $(p^* - p) \cdot \sum_{i=1}^m (y_i^*(p^*) - e_i[\underline{\boldsymbol{\theta}}_0]) \geqslant 0$. Second, there exists $(z_i') \in \Pi_{i=1}^m Z_i^o$ such that $\sum_{i=1}^m z_i^* = \sum_{i=1}^m z_i'$. Henceforth, we let $z := (z_i) := ([z_i^* - z_i']) \in (\mathbb{R}^J)^m$, which satisfies $\sum_{i=1}^m z_i = 0$ and, for each $i \in I$, $W(\underline{\boldsymbol{\theta}}, q^*) z_i^* = W(\underline{\boldsymbol{\theta}}, q^*) z_i$ (since $q^* \in Z_i^{o\perp}$ and $z_i' \in Z_i^o$).

Claim 8 $y^* := (y_i^*) \in \mathcal{A}(p^*)$.

Proof Assume, by contraposition, that $\sum_{i=1}^m (y_i^*(p^*[s])\text{-}e_i[s]) \neq 0$ for some $s \in \underline{\boldsymbol{\theta}}_0$. From Remark 2, $p^*[s] = \sum_{i=1}^m (y_i^*(p^*[s])\text{-}e_i[s]) / \| \sum_{i=1}^m (y_i^*(p^*[s])\text{-}e_i[s])\|$, hence, $\gamma_{(p^*,q^*)}[s] = 0$ and $p^*[s] \cdot \sum_{i=1}^m (y_i^*(p^*[s])\text{-}e_i[s]) > 0$. From Claim 6-($ii$), $p^*[s]\cdot(y_i^*(p^*[s])\text{-}e_i[s]) \leqslant W(q^*)[s]\cdot z_i^*$, for each $i \in I$, hence, from Remark 2, $p^*[s]\cdot \sum_{i=1}^m (y_i^*(p^*[s])\text{-}e_i[s]) \leqslant W(q^*)[s]\cdot \sum_{i=1}^m z_i = 0$, that is, the desired contradiction (with $p^*[s] \cdot \sum_{i=1}^m (y_i^*(p^*[s])\text{-}e_i[s]) > 0$ above). □

Claim 9 *For each* $i \in I$, (y_i^*, z_i^*) *is optimal in* $B_i'(p^*, q^*)$.

Proof Let $i \in I$ and $(y_i', z_i') \in B_i''(p^*, q^*) \subset B_i'(p^*, q^*)$, along Claim 2, be given. From Claim 6-(ii), $(y_i^*, z_i^*) \in B_i'(p^*, q^*)$ and assume, by contraposition, that there exists $(y_i, z_i) \in B_i'(p^*, q^*) \cap \Gamma^*_{(i,p^*)}(y^*) \times Z_i^*$. For every $k \in \mathbb{N}^*$, the convexity of $B_i'(p^*, q^*)$ yields $(y_i^k, z_i^k) := [\frac{1}{k}(y_i', z_i') + (1 - \frac{1}{k})(y_i, z_i)] \in B_i'(p^*, q^*)$, whereas $(y_i^k, z_i^k) \in B_i''(p^*, q^*)$ by construction. Fom Assumptions *A2*, $\Gamma^*_{(i,p^*)}(y^*)$ is open, hence, for some $K > 0$, $y_i^K \in \Gamma^*_{(i,p^*)}(y^*)$, i.e., $(y_i^K, z_i^K) \in B_i''(p^*, q^*) \cap \Gamma^*_{(i,p^*)}(y^*) \times Z_i^*$, contradicting Claim 6-(ii). □

Claim 10 $(y^*, z^*) \in \overline{\mathcal{A}}(p^*, q^*)$, *hence,* $\|y^*(p^*)\| < r$.

Proof Recalling the definitions of sub-Section 3.1, Claim 6-(ii) yields, for every $i \in I$: $(y_i^*, z_i^*) \in B_i'(p^*, q^*) \subset \overline{B}_i(p^*, q^*)$. Hence, from Claims 7-8, $(y^*, z^*) \in \overline{\mathcal{A}}(p^*, q^*)$ and, from Lemma 1 and the choice of r in sub-Section 3.1, $\|y^*(p^*)\| < r$. □

Claim 11 $\gamma_{(p^*,q^*)} = 0$.

Proof Let $(i, s) \in I \times \underline{\boldsymbol{\theta}}_0$ be given. From Definition 2, $p^*[s] \in \mathcal{P}_{(i,p^*,s)}$. We show first that $p^*[s] \cdot (y_i^*(p^*[s]) - e_i[s]) = (W(q^*)z_i^* + \gamma_{(p^*,q^*)})[s]$. Indeed, from Claim 6-($ii$), $p^*[s]\cdot(y_i^*(p^*[s]) - e_i[s]) \leqslant (W(q^*)z_i^* + \gamma_{(p^*,q^*)})[s]$, and, from Claim 10, $\|y_i^*(p^*[s])\| < r$. If $p^*[s]\cdot(y_i^*(p^*[s]) - e_i[s]) < (W(q^*)z_i^* + \gamma_{(p_i,q^*)})[s]$, there exists $y_i \in Y_i^*$, such that $y_i(p^*[s]) > y_i^*(p^*[s])$ and $y_i(\overline{p}[\overline{s}]) = y_i^*(\overline{p}[\overline{s}])$ for every pair $(\overline{p}, \overline{s}) \in \mathcal{P}_{(i,p^*)} \times \theta_0^i$, such that $(\overline{s}, \overline{p}[\overline{s}]) \neq$

$(s, p^*[s])$. From Assumption *A1*, $y_i \in \Gamma^*_{(i,p^*)}(y^*)$ and we may take y_i sufficiently close to y_i^* so that $(y_i, z_i^*) \in B'(p^*, q^*)$. Then, $(y_i, z_i^*) \in B'(p^*, q^*) \cap \Gamma^*_{(i,p^*)}(y^*) \times Z_i^*$, which contradicts Claim 9. Hence, from Remark 2, $p^*[s] \cdot (y_i^*(p^*[s]) - e_i[s]) = W(q^*)[s] \cdot z_i^* + \gamma_{(p^*,q^*)}[s] = W(q^*)[s] \cdot z_i + \gamma_{(p^*,q^*)}[s]$, for each $(i,s) \in I \times \underline{\boldsymbol{\theta}}_0$, with $\sum_{i=1}^m z_i = 0$. Summing up on $i \in I$ these equalities for each $s \in \underline{\boldsymbol{\theta}}_0$ yields, from Claim 8 and the definition, $\gamma_{(p^*,q^*)} = 0$. □

Claim 12 $p^* \in \Delta \subset M_{\underline{\varepsilon}}$*, for the bound* $\underline{\varepsilon}$ *set under Lemma 2, hence,* $\|y^*\| + \|z^*\| < r$.

Proof We let the reader check, from Claims 9 and 10 and from Assumption *A1*, by a similar contradiction argument as in the proof of Claim 11, that $p^* \geqslant 0$. Then, from Claim 11, $p^* \in \mathcal{M}^*$. Let $s \in \underline{\boldsymbol{\theta}}$ be given. From Remark 2, $(z_i) \in \mathcal{Z}$ and there exists $i \in I$, such that $V[s] \cdot z_i^* = V[s] \cdot z_i \geqslant 0$. By the same token, we let the reader check, from Claims 9 and 10 and from Assumption *A3*, that (i, p^*, y^*) meets the condition of the definition of Δ_s. Hence, $p^* \in \Delta_s$ and, from Lemma 2, $p^* \in \Delta := \cap_{s \in \underline{\boldsymbol{\theta}}} \Delta_s \subset \mathcal{M}_{\underline{\varepsilon}}$. From Claim 10, Lemma 1 and the choice of r in sub-Section 3.1, $\|y^*\| + \|z^*\| < r$. □

Claim 13 *Let* $z \in (\mathbb{R}^J)^m$ *be defined as in Remark 2. Then,* $((p^*, q^*), (y^*, z))$ *is a C.F.E., which satisfies the (price and no-arbitrage) Conditions of Theorem 1.*

Proof Let $\mathcal{C} := ((p^*, q^*), (y^*, z))$ be defined as in Claim 13. From Claim 1, if $\mathcal{C}$ is a C.F.E., it meets the price and no-arbitrage Conditions of Theorem 1.

By construction, from Remark 2 and Claims 6, 8 and 11, $(y^*, z) \in \Pi_{i=1}^m B_i(p^*, q^*)$ and $\mathcal{C}$ satisfies Conditions (b) and (c) of Definition 3 of equilibrium. Assume, by contraposition, that $\mathcal{C}$ fails to meet Condition (a) of Definition 3, that is, there exists $i \in I$, say $i = 1$, and $(\overline{y}_1, \overline{z}_1) \in B_1(p^*, q^*) \cap \Gamma_{(1,p^*)}(y_1^*) \times \mathbb{R}^J$. Let $\widetilde{z}_1 \in Z_1^{o\perp}$ be the orthogonal projection of $\overline{z}_1$ on $Z_1^{o\perp}$. Then, $(\overline{y}_1, \widetilde{z}_1) \in B_1(p^*, q^*) \cap Y_1 \times Z_1^{o\perp}$. From Assumption *A3*, Claim 12 and the convexity of $B_1(p^*, q^*) \cap Y_1 \times Z_1^{o\perp}$, we may take $(\overline{y}_1, \overline{z}_1)$ so close to

(y_1^*, z_1^*) that $(\overline{y}_1, \widetilde{z}_1) \in B_1'(p^*, q^*)$. Then, $(\overline{y}_1, \widetilde{z}_1) \in B_1'(p^*, q^*) \cap \Gamma_{(1,p^*)}(y^*) \times \mathbb{R}^J$, contradicting Claims 9-11. Hence, $\mathcal{C}$ also meets Condition (a) of Definition 3, i.e., $\mathcal{C}$ is a C.F.E. $\square$

Appendix

We recall the notations of sub-Section 3.1, in particular:

$\overline{B}_i(p,q) := \{(y,z) \in Y_i \times Z_i^{o\perp} : \overline{p}[s] \cdot (y(\overline{p}[s]) - e_i[s]) \leqslant W(q)[s] \cdot z + 1, \forall (\overline{p}, s) \in \mathcal{P}_{(i,p)} \times \theta_0^i\}$;

$\overline{\mathcal{A}}(p,q) := \{[(y_i, z_i)] \in \Pi_{i=1}^m \overline{B}_i(p,q) : (y_i) \in \mathcal{A}(p), \sum_{i=1}^m z_i \in Z^o\}$.

Lemma 1 *The following Assertions hold:*

(i) $\exists r > 0 : (p \in \mathcal{M}$ *and* $y \in \mathcal{A}(p)) \Rightarrow \|y(p)\| < r$;

(ii) $\forall \varepsilon \in]0, \frac{1}{L}[, \exists r_\varepsilon > 0 : ((p,q) \in \Pi_\varepsilon,\ (y,z) \in \overline{\mathcal{A}}(p,q)) \Rightarrow (\|y\| + \|z\| < r_\varepsilon)$.

Proof

(i) We recall that, for every $(i,p) \in I \times \mathcal{M}$, $y_i(p) := (y_i(p[s]))_{s \in \underline{\boldsymbol{\theta}}_0}$. We denote $\alpha := \|(e_i)\|$ and $r := 1 + m\alpha$, and let $p \in \mathcal{M}$ and $(y_i) \in \mathcal{A}(p)$ be given. For each $i \in I$, $X_i = \mathbb{R}_+^{L\theta_0^i}$ and $(y_i) \in \mathcal{A}(p)$ imply $0 \leqslant y_i(p) \leqslant \sum_{j \in I} e_j[\underline{\boldsymbol{\theta}}_0]$, hence, $\|y_i(p)\| \leqslant \alpha$ and $\|(y_i(p))\| < r$. This proves Lemma 1-(i).

(ii) Let $\varepsilon \in]0, \frac{1}{L}[$ be given. The proof of Lemma-(ii) proceeds in steps.

- First, we show: $\forall M > 0, \exists r_\varepsilon^M > 0 : [(p,q) \in \Pi_\varepsilon,\ (y,z) \in \overline{\mathcal{A}}(p,q),\ \|z\| < M] \Rightarrow \|y\| < r_\varepsilon^M$.

We let $\delta' := \delta\varepsilon$, where $\delta > 0$ satisfies the condition of Assumption *B2*, $\beta := \|(V[s])_{s \in S}\|$ and set as given $M > 0$, $(p,q) \in \Pi_\varepsilon$, $(y,z) := [(y_i, z_i)] \in \overline{\mathcal{A}}(p,q)$, such that $\|z\| < M$. We let the reader check, from the definition of Π_ε, from Assumption *B2* and from above that: $0 \leqslant y_i^l(\overline{p}[s]) \leqslant \gamma_\varepsilon^M := r + \alpha + [1 + \beta M J]/\delta'$, for every $(i,l) \in I \times \{1, ..., L\}$ and every $(\overline{p}, s) \in \mathcal{P}_{(i,p)} \times \theta_0^i$. Then, $\|y\| := \|(y_i)\| < r_\varepsilon^M := \gamma_\varepsilon^M L n m^2$, where $n := \otimes_{i \in I}(\#\mathcal{P}_{(i,p)})$.

- Second: $\exists r^2_\varepsilon > 0 : [(p,q) \in \Pi_\varepsilon \ \mathit{and} \ (y,z) \in \overline{\mathcal{A}}(p,q)] \Rightarrow \|z\| < r^2_\varepsilon$

Assume, by contraposition, that, for every positive integer n, there exist $(p^n, q^n) \in \Pi_\varepsilon$ and $(y^n, z^n) := [(y^n_i, z^n_i)] \in \overline{\mathcal{A}}(p^n, q^n)$, such that $\|z^n\| > n$. Each set $\overline{\mathcal{A}}(p^n, q^n)$ (for $n > 0$) is closed and convex and contains (y^n, z^n) and $(y^e, 0) := [(y^{e_i}, 0)]$, where, for each $i \in I$, y^{e_i} is defined by $y^{e_i}(\overline{p}) := e_i$, for each $\overline{p} \in \mathcal{P}_{(i,p^n)}$. For every $(i,n) \in I \times \mathbb{N}^*$, we let $y'^n_i := \frac{y^n_i}{\|z^n\|} + (1 - \frac{1}{\|z^n\|})y^{e_i}$ and $z'^n_i := \frac{z^n_i}{\|z^n\|}$, hence, $(y'^n, z'^n) := [(y'^n_i, z'^n_i)] \in \overline{\mathcal{A}}(p^n, q^n)$ and $\|z'^n\| = 1$. From above, the sequences $\{(y'^n, z'^n)\}_{n \geqslant 1}$ and $\{(p^n, q^n)\}_{n \geqslant 1}$ are bounded in compact sets and may be assumed to converge, say, towards (y', z') and $(p', q') \in \Pi_\varepsilon$. Moreover, the continuity of the scalar product and the fact that correspondence $(p,q) \mapsto \overline{\mathcal{A}}(p,q)$ is closed yield $\|z'\| = 1$ and $(y', z') \in \overline{\mathcal{A}}(p', q')$.

Given $N > 1$, for all $n > N$, we let $y^n_N := (\frac{N}{\|z_n\|} y^n_i + (1 - \frac{N}{\|z_n\|})y^{e_i})_{i \in I} \in Y$. By the same token, for $n > N$, $(y^n_N, Nz'^n) \in \overline{\mathcal{A}}(p^n, q^n)$ and the bounded sequence $[(y^n_N, Nz'^n)]_{n > N}$ may be assumed to converge to some $(y_N, z_N) \in \overline{\mathcal{A}}(p', q')$, such that $z_N = Nz'$ (for the same z' and (p', q') as above). Hence, for every $N > 1$, there exists $(y_N, z_N) \in \overline{\mathcal{A}}(p', q')$, such that $z_N = Nz'$, with $\|z'\| = 1$.

All consumption sets being bounded from below, this latter condition implies (from step 1 of this proof and from Assumption *B2*) $V(\theta^i)z'_i \geqslant 0$, for every $i \in I$, whereas $\sum_{i=1}^m z'_i \in Z^o$ (since $(y', z') \in \overline{\mathcal{A}}(p', q')$) and $[V, \theta]$ is arbitrage-free. Then, it results from [3, Proposition 3.1, p. 401] that $z' := (z'_i) \in \Pi_{i=1}^m Z^o_i$, whereas the relation $(y', z') \in \overline{\mathcal{A}}(p', q')$ implies $z' \in \Pi_{i=1}^m Z_i^{o\perp}$ from the definition. Thus, $z' \in \Pi_{i=1}^m Z^o_i \cap \Pi_{i=1}^m Z_i^{o\perp} = \{0\}$, which contradicts the condition $\|z'\| = 1$. Hence, there exists $r^2_\varepsilon > 0$, such that: $[(p,q) \in \Pi_\varepsilon,\ (y,z) \in \overline{\mathcal{A}}(p,q)] \Rightarrow \|z\| < r^2_\varepsilon$.

- Third, from above, Lemma 1-(ii) holds for $r_\varepsilon = M + r^M_\varepsilon$, with $M = r^2_\varepsilon$. □

We now prove Lemma 2.

Lemma 2 *The following Assertions hold:*

(i) *For every* $s \in \underline{\boldsymbol{\theta}}$, Δ_s *is compact, hence* Δ *is a compact set;*

(ii) *There exists* $\underline{\varepsilon} \in]0, \frac{1}{L}[$, *such that* $\Delta \subset \mathcal{M}_{\underline{\varepsilon}}$.

We recall that $\Delta := \cap_{s \in \underline{\boldsymbol{\theta}}} \Delta_s$, where, for each $s \in \underline{\boldsymbol{\theta}}$:

$\Delta_s := \{p \in \mathcal{M}^* : \exists i \in I, \exists y := (y_i) \in \mathcal{A}(p)$, *such that:* $(x_i \in \Gamma_{(i,p)}(y_i)$, $x_i(\overline{p}[\overline{s}]) = y_i(\overline{p}[\overline{s}])$, *for every* $(\overline{p}, \overline{s}) \in \mathcal{P}_{(i,p)} \times \theta_0^i \backslash \{s\}$ *and* $x_i(\overline{p}[s]) = y_i(\overline{p}[s])$, *for every* $\overline{p}[s] \in \mathcal{P}_{(i,p,s)} \backslash \{p[s]\})$ *imply* $(p[s] \cdot x_i(p[s]) \geqslant p[s] \cdot y_i(p[s]) \geqslant p[s] \cdot e_i[s])\}$.

Proof

(i) Let $s \in \underline{\boldsymbol{\theta}}$ and a converging sequence $(p^n)_{n \in \mathbb{N}}$ of elements of Δ_s be given (from Claim 12, $\Delta_s \neq \varnothing$). Since $\mathcal{M}^*$ is closed, there exists $p \in \mathcal{M}^*$, s.t. $p := \lim p^n$. Moreover, w.l.o.g., we may assume there exist $i \in I$ and a sequence $(y^n)_{n \in \mathbb{N}} := ((y_i^n))_{n \in \mathbb{N}}$ of $y^n \in \mathcal{A}(p^n)$, such that, for each $n \in \mathbb{N}$, (p^n, i, y^n) satisfies the condition of the definition of Δ_s. From the proof of Lemma 1, $(y^n(p^n)) := ((y_i^n(p^n))_{i \in I})_{n \in \mathbb{N}}$ is bounded. Moreover, we let the reader check, from the Assumption *S2* of separability, that we may assume, for every $(i, n) \in I \times \mathbb{N}$, that $y_i^n(\overline{p}[\overline{s}]) = e_i[\overline{s}]$, for every $(\overline{p}, \overline{s}) \in \mathcal{P}_{(i,p^n)} \times \theta^i \backslash \{s\}$ and $y_i^n(\overline{p}[s]) = e_i[s]$, for every $\overline{p}[s] \in \mathcal{P}_{(i,p^n,s)} \backslash \{p^n[s]\}$. Thus, we also assume at no cost that $(y^n)_{n \in \mathbb{N}}$ converges, say to $y := (y_i) \in \mathcal{A}(p)$.

The conditions $p^n[s] \cdot (y_i^n(p[s]) - e_i[s]) \geqslant 0$ of the definition of Δ_s, for every $n \in \mathbb{N}$, yield, in the limit, $p[s] \cdot (y_i(p[s]) - e_i[s]) \geqslant 0$. We show that (p, i, y) satisfies the condition of the definition of Δ_s (hence, $p := \lim p^n \in \Delta_s$, i.e., Δ_s is closed). If not, from the definition of Δ_s, there exists $x_i \in \Gamma_{(i,p)}(y_i)$, such that $x_i(\overline{p}[\overline{s}]) = y_i(\overline{p}[\overline{s}]) = e_i[\overline{s}]$, for every

$(\overline{p},\overline{s}) \in \mathcal{P}_{(i,p)} \times \theta_0^i \backslash \{s\}$ and $x_i(\overline{p}[s]) = y_i(\overline{p}[s]) = e_i[s]$, for every $\overline{p}[s] \in \mathcal{P}_{(i,p,s)} \backslash \{p[s]\}$ and $p[s] \cdot (x_i(p[s]) - y_i(p[s])) < 0$. Then, $(p, y_i) := \lim(p^n, y_i^n)$ and the fact that the correspondence $(p', y') \mapsto \Gamma_{(i,p')}(y')$ is open (from Assumptions *B1-S2-A2*) imply $x_i \in \Gamma_{(i,p^N)}(y_i^N)$ and $p^N[s] \cdot (x_i(p[s]) - y_i^N(p[s])) < 0$, for $N \in \mathbb{N}$, large enough, which contradicts the fact that (p^N, i, y^N) satisfies the condition of the definition of Δ_s. This contradiction proves that $p \in \Delta_s$, hence, Lemma 2-(i) holds.

(ii) Let $(l, s, p) \in \{1, ..., L\} \times \underline{\boldsymbol{\theta}} \times \Delta$ be given. Then, $p \in \Delta_s$ and we let $(i, y := (y_i), p)$ meet the condition of the definition of Δ_s and $e_{(l,s)} \in Y_i$ be defined by $e^l_{(l,s)}(p[s]) = 1$ and $e_{(l,s)}(\overline{p}[\overline{s}]) = 0$, for every $(\overline{p}, \overline{s}) \in \mathcal{P}_{(i,p)} \times \theta_0^i \backslash \{s\}$, $e_{(l,s)}(\overline{p}[s]) = 0$, for every $\overline{p}[s] \in \mathcal{P}_{(i,p,s)} \backslash \{p[s]\}$ and $e^{\overline{l}}_{(l,s)}(p[s]) = 0$, for every $\overline{l} \in \{1, ..., L\} \backslash \{l\}$.

We prove, first, that $p^l[s] > 0$. From Assumption *S1* and the fact that $\|p[s]\| = 1$, there exists $a_i \in Y_i$, such that $a_i(\overline{p}[\overline{s}]) = y_i(\overline{p}[\overline{s}])$, for all $(\overline{p}, \overline{s}) \in \mathcal{P}_{(i,p)} \times \theta_0^i \backslash \{s\}$, $a_i(\overline{p}[s]) = y_i(\overline{p}[s])$, for all $\overline{p}[s] \in \mathcal{P}_{(i,ps)} \backslash \{p[s]\}$ and $p[s] \cdot a_i(p[s]) < p[s] \cdot e_i[s] \leqslant p[s] \cdot y_i(p[s])$.

Then, for all $n \in \mathbb{N}^*$, $x_i^n := (\frac{1}{n} a_i + (1 - \frac{1}{n}) y_i) \in Y_i$ satisfies $p[s] \cdot x_i^n(p[s]) < p[s] \cdot y_i(p[s])$ by construction. From Assumptions *A1-A2*, $x^N := (x_i^N + (1 - \frac{1}{N}) e_{(l,s)}) \in \Gamma_{(i,p)}(y_i)$, for $N \in \mathbb{N}^*$ big enough, which implies, since (i, y, p) meets the condition of the definition Δ_s and from above, $p[s] \cdot y_i(p[s]) \leqslant p[s] \cdot x^N(p[s]) < p[s] \cdot y_i(p[s]) + (1 - \frac{1}{N}) p^l[s]$, that is, $p^l[s] > 0$.

Hence, for every $p \in \Delta_s$ and every $l \in \{1, ..., L\}$, there exists $\varepsilon_{(p,l,s)} \in]0, \frac{1}{L}[$, such that $p^l[s] > \varepsilon_{(p,l,s)}$. The mapping $\varphi_{(l,s)} : \Delta_s \to \mathbb{R}_{++}$, defined by $\varphi_{(l,s)}(p) := p^l[s]$ (for $(p, l) \in \Delta_s \times \{1, ..., L\}$) is continuous and attain its minimum for some $p_{(l,s)}$ in the compact set Δ_s. We let the reader check that $\underline{\varepsilon} := \min \varepsilon_{(p_{(l,s)}, l, s)}$, for $(l, s) \in \{1, ..., L\} \times \underline{\boldsymbol{\theta}}$, satisfies $\Delta \subset \mathcal{M}_{\underline{\varepsilon}}$. $\square$

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